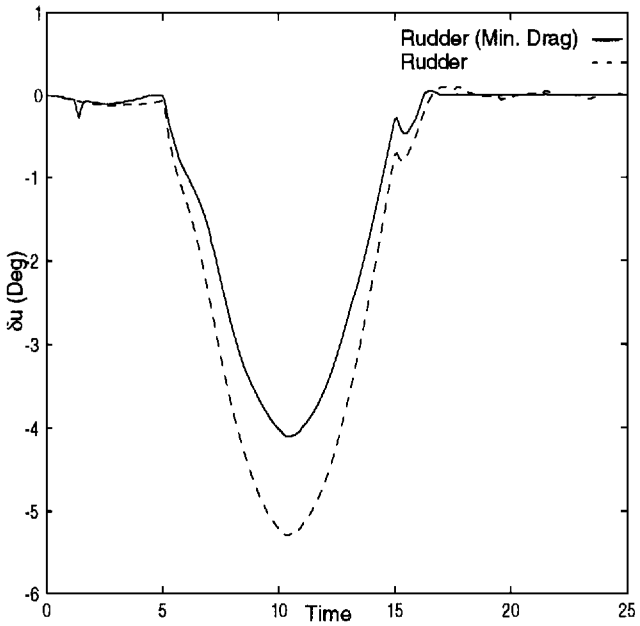

Fig. 5 Rudder time histories.

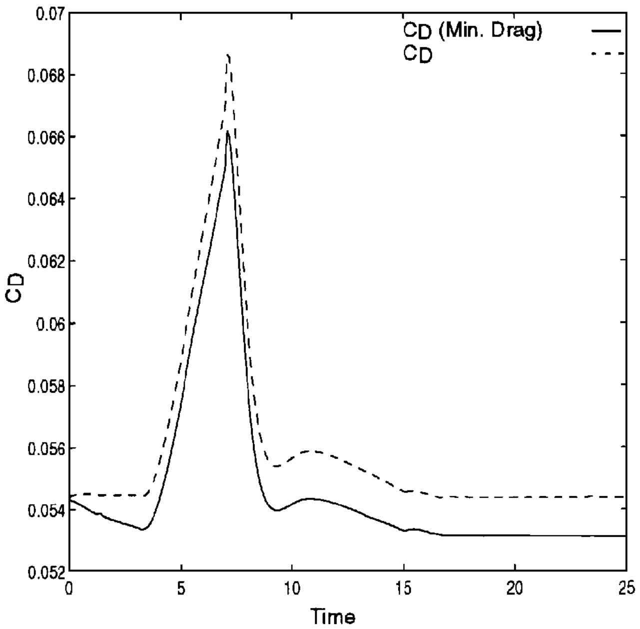

Fig. 6 Drag coefficient time histories.

toward minimum drag. Finally, Fig. 6 compares the drag coefficients for the pseudoinverse and minimum drag allocation schemes during the maneuver. As expected, the minimum drag results are less throughout the entire maneuver. When the steady-state conditions are reached, the control deflections given by the minimum drag allocation scheme result in a 2.4% reduction in drag.

Conclusions

The example has demonstrated that whereas direct control allocation itself presents a powerful approach to solving the three-moment, multiple control problem, it can also be modified, with relative ease, to account for a larger four-objective, multiple control problem. In this case, drag was considered a fourth objective, which the allocator was instructed to minimize. The minimization procedure here is not the usual iterative/optimization algorithm; it simply allocates toward this state. Whether it settles on a minimum drag state or not depends primarily on the tasks that the aircraft is performing. In the case of extended periods of inactivity where the control laws are not demanding extreme control deflections (like trimmed flight), the allocation procedure does in fact achieve a minimum drag configuration.

Acknowledgments

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Noetherian Perspective of Eulerian Motion of a Free Rigid Body

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Introduction

CONSERVATION laws play an important role in mechanics from both a theoretical and a practical standpoint. They can considerably simplify the integration of the differential equations of motion. Also, they can be regarded as the manifestation of some profound physical principle. Although there are a variety of approaches for finding conservation laws, the most popular and modern method is based on the study of the invariant properties of the Lagrangian with respect to infinitesimal transformations of the generalized coordinates describing the configuration of the system and time. This approach is based on Noether's famous theorem.[1] Noether's theory can be regarded as a generalization of the theory of ignorable coordinates[1] although it is not usually treated from this viewpoint.

There exists a conservation law for conservative scleronomic systems, or for mechanical systems whose Lagrangian functions are not explicit functions of time. This conservation law is identical to the conservation of energy if the kinetic energy of the system can be expressed in a form quadratic in the generalized velocities. The conservation law can be obtained by redefining time as one of the generalized coordinates and using the theory of ignorable coordinates[1] or by applying Noether's theorem after an infinitesimal transformation of time.[1]

It is well known that the linear momentum of a rigid body is a conserved quantity in the absence of external forces. This conservation law can be obtained from the theory of ignorable coordinates[1] by observing that the kinetic energy of the body is not a function of the coordinates of the center of mass. Linear momentum conservation can also be established by using Noether's theorem[1] after an infinitesimal translation of the coordinate system.

The angular momentum of a free rigid body (rigid body in the absence of external forces and moments) is a conserved quantity. If Euler angles are used for the description of orientation of the

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rigid body, however, this result cannot be obtained using the theory of ignorable coordinates because Euler angles are not ignorable coordinates. When the kinetic energy of the rigid body is expressed in terms of its Eulerian motion, the derivation of the conservation of angular momentum from the application of Noether's theorem is also not straightforward. As we address this problem, we arrive at a new identity involving the Euler angles and their rates. Our discussion is relevant to the theory of quasicoordinates,[2] and the identity obtained can be substituted into Lagrange's equations to establish angular momentum conservation.

Background

Consider a rigid body rotating about its center of mass described with the help of two reference frames: the inertially fixed reference frame xyz and the body-fixed reference frame $x'''y'''z'''$. The origins of both the frames are assumed to be coincident and fixed to the center of mass of the rigid body. The orientation of the body-fixed reference frame $x'''y'''z'''$ with respect to the reference frame xyz is expressed with the help of three Euler angles μ_1, μ_2, and μ_3. The particular set of Euler angles used is not specified here for the sake of generality. For the purpose of visualization, however, z-y-x Euler angles (α, β, γ) are used in Fig. 1 to describe the orientation of $x'''y'''z'''$ with respect to xyz.

Let $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_B$ represent the angular velocity of the rigid body in the inertially fixed reference frame xyz and in the body-fixed reference frame $x'''y'''z'''$, respectively, and let $\boldsymbol{\mu} \triangleq (\mu_1 \;\; \mu_2 \;\; \mu_3)^T$ denote the vector of Euler angles. Let $\boldsymbol{R}(\boldsymbol{\mu})$ denote the orthogonal transformation matrix between the $x'''y'''z'''$ and xyz reference frames. Then, the following relations hold:

$$\boldsymbol{\omega}_I = \boldsymbol{R}(\boldsymbol{\mu})\,\boldsymbol{\omega}_B, \qquad \left(\frac{\partial \boldsymbol{\omega}_B}{\partial \boldsymbol{\omega}_I}\right) = \boldsymbol{R}^T \tag{1}$$

Also, the Euler angle rates, irrespective of whichever set of angles we prefer to use, are related to the angular velocities $\boldsymbol{\omega}_I$ and $\boldsymbol{\omega}_B$ according to relations of the form

$$\boldsymbol{\omega}_I = \boldsymbol{K}(\boldsymbol{\mu})\dot{\boldsymbol{\mu}}, \qquad \boldsymbol{\omega}_B = \boldsymbol{R}^T\boldsymbol{K}(\boldsymbol{\mu})\dot{\boldsymbol{\mu}} \tag{2}$$

In the case of z-y-x Euler angles (α, β, γ), the matrix $\boldsymbol{K}(\alpha, \beta, \gamma)$ has the form

$$\boldsymbol{K}(\alpha, \beta, \gamma) \triangleq \begin{pmatrix} 0 & -\sin\alpha & \cos\alpha\cos\beta \\ 0 & \cos\alpha & \sin\alpha\cos\beta \\ 1 & 0 & -\sin\beta \end{pmatrix} \tag{3}$$

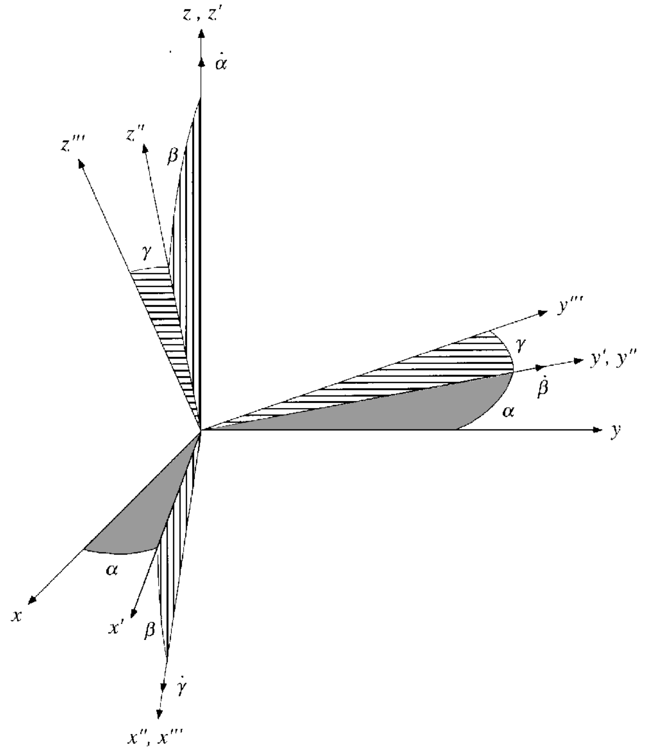

Fig. 1 Description of z-y-x Euler angles (α, β, γ).

The column vectors of the matrix $\boldsymbol{K}(\alpha, \beta, \gamma)$ are unit vectors that provide the direction of the Euler angle rates $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ in the inertial reference frame. For example, $\dot{\alpha}$ has the direction of the unit vector along the z axis. This unit vector, shown in Fig. 1, is the first column vector of the matrix $\boldsymbol{K}$ in Eq. (3) and can be denoted as $\boldsymbol{K}_1$. The Euler angle rates $\dot{\beta}$ and $\dot{\gamma}$ have the directions of the unit vectors along the y' and x'' axes, which form the second and third column vectors of $\boldsymbol{K}$, respectively. These unit vectors can be denoted as $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$, respectively.

Euler Angle Identity

In this section we establish, for the first time, that the identity

$$\boldsymbol{R}^T\dot{\boldsymbol{K}} = \frac{\partial}{\partial \boldsymbol{\mu}}(\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}) \tag{4}$$

is true for any arbitrary set of Euler angles. To prove the identity we rewrite it in the form

$$\dot{\boldsymbol{K}}_i = \boldsymbol{R}\frac{\partial}{\partial \mu_i}(\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}), \qquad i = 1, 2, 3 \tag{5}$$

where $\boldsymbol{K}_i$ represents the ith column vector of the matrix $\boldsymbol{K} \in R^{3\times 3}$. It can be easily shown that

$$\frac{\partial}{\partial \mu_i}(\boldsymbol{K}\dot{\boldsymbol{\mu}}) = \frac{\partial}{\partial \mu_i}(\boldsymbol{R}\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}) = \boldsymbol{R}\frac{\partial}{\partial \mu_i}(\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}) + \left(\frac{\partial \boldsymbol{R}}{\partial \mu_i}\right)\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}$$

Therefore, it follows that

$$\begin{aligned} \boldsymbol{R}\frac{\partial}{\partial \mu_i}(\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}) &= \frac{\partial}{\partial \mu_i}(\boldsymbol{K}\dot{\boldsymbol{\mu}}) - \left(\frac{\partial \boldsymbol{R}}{\partial \mu_i}\right)\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}} \\ &= \left(\frac{\partial \boldsymbol{\omega}_I}{\partial \mu_i}\right) - \left(\frac{\partial \boldsymbol{R}}{\partial \mu_i}\right)\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}} \end{aligned} \tag{6}$$

From Eqs. (5) and (6) it now follows that Eq. (4) can be established by proving

$$\dot{\boldsymbol{K}}_i = \left(\frac{\partial \boldsymbol{\omega}_I}{\partial \mu_i}\right) - \left(\frac{\partial \boldsymbol{R}}{\partial \mu_i}\right)\boldsymbol{R}^T\boldsymbol{K}\dot{\boldsymbol{\mu}}, \qquad i = 1, 2, 3 \tag{7}$$

The left-hand side of Eq. (7) can be explicitly written as

$$\dot{\boldsymbol{K}}_1 = 0 \tag{8a}$$

$$\dot{\boldsymbol{K}}_2 = \dot{\mu}_1\boldsymbol{K}_1 \times \boldsymbol{K}_2 \tag{8b}$$

$$\dot{\boldsymbol{K}}_3 = (\dot{\mu}_1\boldsymbol{K}_1 + \dot{\mu}_2\boldsymbol{K}_2) \times \boldsymbol{K}_3 \tag{8c}$$

Equations (8a–8c) can be explained as follows. The Euler angles μ_1, μ_2, and μ_3 represent a sequence of three rotations about the unit vectors $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, and $\boldsymbol{K}_3$, respectively. From the definition of Euler angles, the unit vector $\boldsymbol{K}_1$ is directed along one of the three axes, x, y, or z. For example, in the case of z-y-x Euler angles the unit vector $\boldsymbol{K}_1$ is directed along the z axis. Therefore, $\boldsymbol{K}_1$ is a constant, and $\dot{\boldsymbol{K}}_1 = 0$. The unit vector $\boldsymbol{K}_2$ denotes the second axis of rotation in the Euler angle sequence. Generally speaking, the direction of $\boldsymbol{K}_2$ is coincident with one of the three axes, x', y', or z', where x', y', and z' are the new directions of x, y, and z after a rotation of μ_1 about $\boldsymbol{K}_1$. The angular velocity of $\boldsymbol{K}_2$ is, therefore, $\dot{\mu}_1\boldsymbol{K}_1$ and, hence, Eq. (8b) follows. In the case of z-y-x Euler angles (α, β, γ), shown in Fig. 1, $\boldsymbol{K}_2$ has the same direction as the y' axis where y' is obtained from y after a rotation of α about $\boldsymbol{K}_1$ or the z axis. The unit vector $\boldsymbol{K}_3$ denotes the third axis of rotation and, generally speaking, it is coincident with one of the three axes, x'', y'', or z'', where x'', y'', and z'' are the new directions of x, y, and z after a rotation of μ_1 about $\boldsymbol{K}_1$ followed by a rotation of μ_2 about $\boldsymbol{K}_2$. The angular velocity of $\boldsymbol{K}_3$ can, therefore, be expressed as $(\dot{\mu}_1\boldsymbol{K}_1 + \dot{\mu}_2\boldsymbol{K}_2)$ from which Eq. (8c) follows. For the z-y-x Euler angles (α, β, γ), shown in Fig. 1, $\boldsymbol{K}_3$ has the same direction as the x'' axis where x'' is obtained from x after a rotation of α about $\boldsymbol{K}_1$ or z followed by a rotation of β about $\boldsymbol{K}_2$ or y'.

The first term in the right-hand side of Eq. (7) can be simplified by first writing the expression for the angular velocity vector ω_I, from Eq. (2), as

$$\omega_I = \dot{\mu}_1 K_1 + \dot{\mu}_2 K_2 + \dot{\mu}_3 K_3 \tag{9}$$

Clearly, ω_I is a weighted sum of the three unit vectors K_1, K_2, and K_3. Among these, K_1 has a fixed direction, the direction of K_2 depends on the Euler angle μ_1, and the direction of K_3 depends on the Euler angles μ_1 and μ_2. Therefore, a change in the Euler angle μ_1 results in a change in the unit vectors K_2 and K_3, a change in μ_2 results in a change in the unit vector K_3, and a change in μ_3 produces no change in any one of the three unit vectors. Hence, the first term in the right-hand side of Eq. (7) can be written explicitly as

$$\left(\frac{\partial \omega_I}{\partial \mu_1}\right) = \frac{\partial}{\partial \mu_1}(\dot{\mu}_2 K_2 + \dot{\mu}_3 K_3) = K_1 \times (\dot{\mu}_2 K_2 + \dot{\mu}_3 K_3) \tag{10a}$$

$$\left(\frac{\partial \omega_I}{\partial \mu_2}\right) = \frac{\partial}{\partial \mu_2}(\dot{\mu}_3 K_3) = K_2 \times \dot{\mu}_3 K_3 \tag{10b}$$

$$\left(\frac{\partial \omega_I}{\partial \mu_3}\right) = 0 \tag{10c}$$

Equation (10c) can be quickly verified for the z-y-x Euler angles (α, β, γ) by observing that the matrix K in Eq. (3) is not a function of the Euler angle γ.

The second term in the right-hand side of Eq. (7) can be written as

$$\left(\frac{\partial R}{\partial \mu_i}\right) = (K_i \times R_1 \quad K_i \times R_2 \quad K_i \times R_3) \tag{11}$$

where R_j is the jth column vector of the matrix R. By substituting Eq. (11), the second term in the right-hand side of Eq. (7) now takes the form

$$\begin{aligned} -\left(\frac{\partial R}{\partial \mu_i}\right) R^T K \dot{\mu} &= -(K_i \times R_1 \quad K_i \times R_2 \quad K_i \times R_3) R^T K \dot{\mu} \\ &= -(K_i \times) R R^T K \dot{\mu} \\ &= -(K_i \times) K \dot{\mu} \\ &= (K_1 \times K_i \quad K_2 \times K_i \quad K_3 \times K_i) \dot{\mu} \end{aligned} \tag{12}$$

where $(K_i \times)$ represents the skew-symmetric matrix operator for a rotation along the unit vector K_i. Equation (12) can be explicitly written as

$$-\left(\frac{\partial R}{\partial \mu_1}\right) R^T K \dot{\mu} = \dot{\mu}_2 K_2 \times K_1 + \dot{\mu}_3 K_3 \times K_1 \tag{13a}$$

$$-\left(\frac{\partial R}{\partial \mu_2}\right) R^T K \dot{\mu} = \dot{\mu}_1 K_1 \times K_2 + \dot{\mu}_3 K_3 \times K_2 \tag{13b}$$

$$-\left(\frac{\partial R}{\partial \mu_3}\right) R^T K \dot{\mu} = \dot{\mu}_1 K_1 \times K_3 + \dot{\mu}_2 K_2 \times K_3 \tag{13c}$$

The identity in Eq. (4) or Eq. (7) can now be established from Eqs. (8), (10), and (13).

Noetherian Perspective of Free Eulerian Motion

In simple words, Noether's principle states that any infinitesimal transformation of the position coordinates that leave the Lagrangian unchanged leads automatically to a certain conservation law.[1] Under zero external forces and moments, the Lagrangian of a free rigid body is equivalent to the kinetic energy of the body and is given by the relation

$$L = \tfrac{1}{2} M v_c^T v_c + \tfrac{1}{2} \omega_B^T I_B \omega_B \tag{14}$$

where M and v_c are the total mass and the velocity of the center of mass of the rigid body, I_B represents the inertia matrix in the body fixed reference frame and is a constant, and ω_B is the angular velocity of the body and is a function of the Euler angles and their rates. A change in the Lagrangian with respect to the orientation and the orientation rate can now be written as

$$\begin{aligned} \mathrm{d}L &= \left(\frac{\partial L}{\partial \mu}\right) \mathrm{d}\mu + \left(\frac{\partial L}{\partial \dot{\mu}}\right) \mathrm{d}\dot{\mu} \\ &= \left(\frac{\partial L}{\partial \omega_B}\right)\left(\frac{\partial \omega_B}{\partial \mu}\right) \mathrm{d}\mu + \left(\frac{\partial L}{\partial \omega_B}\right)\left(\frac{\partial \omega_B}{\partial \dot{\mu}}\right) \mathrm{d}\dot{\mu} \\ &= \left(\frac{\partial L}{\partial \omega_B}\right)\left[\frac{\partial}{\partial \mu}(R^T K \dot{\mu})\, \mathrm{d}\mu + R^T K\, \mathrm{d}\dot{\mu}\right] \end{aligned} \tag{15}$$

where the expression $\omega_B = R^T K \dot{\mu}$ was substituted from Eq. (2). From Eq. (2) we can additionally write

$$\mathrm{d}\omega_I = K\, \mathrm{d}\dot{\mu} + \dot{K}\, \mathrm{d}t\, \dot{\mu} = K\, \mathrm{d}\dot{\mu} + \dot{K}\, \mathrm{d}\mu \tag{16}$$

By manipulating Eq. (15) and substituting Eqs. (1), (4), and (16), we get

$$\begin{aligned} \mathrm{d}L &= \left(\frac{\partial L}{\partial \omega_B}\right)\left[\frac{\partial}{\partial \mu}(R^T K \dot{\mu})\, \mathrm{d}\mu + R^T (\mathrm{d}\omega_I - \dot{K}\, \mathrm{d}\mu)\right] \\ &= \left(\frac{\partial L}{\partial \omega_B}\right)\left[\frac{\partial}{\partial \mu}(R^T K \dot{\mu}) - R^T \dot{K}\right] \mathrm{d}\mu + \left(\frac{\partial L}{\partial \omega_B}\right) R^T\, \mathrm{d}\omega_I \\ &= \left(\frac{\partial L}{\partial \omega_B}\right) \mathrm{d}\omega_B \end{aligned} \tag{17}$$

Equation (17) implies that the Lagrangian of a free rigid body remains invariant for infinitesimal changes in the orientation of the body. On the basis of Noether's theorem this implies that there exists a conserved quantity. In the next section we show that the conserved quantity is indeed the angular momentum of the rigid body.

Conservation of Angular Momentum

The Lagrangian of a free rigid body is equivalent to the kinetic energy of the body and is given by the relationships

$$\begin{aligned} L &= \tfrac{1}{2} M v_c^T v_c + \tfrac{1}{2} \omega_B^T I_B \omega_B = \tfrac{1}{2} M v_c^T v_c + \tfrac{1}{2} \omega_I^T I_I \omega_I \\ I_I &\triangleq R I_B R^T \end{aligned} \tag{18}$$

where I_B and I_I are the inertia matrices of the rigid body in the body-fixed reference frame and the inertial reference frame, respectively. From Eq. (18) it can be shown that

$$\left(\frac{\partial L}{\partial \dot{\mu}}\right) = \left(\frac{\partial L}{\partial \omega_B}\right)\left(\frac{\partial \omega_B}{\partial \dot{\mu}}\right) = \omega_B^T I_B R^T K = \omega_I^T I_I K = H_I^T K \tag{19}$$

and

$$\begin{aligned} \left(\frac{\partial L}{\partial \mu}\right) &= \left(\frac{\partial L}{\partial \omega_B}\right)\left(\frac{\partial \omega_B}{\partial \mu}\right) = \omega_B^T I_B \frac{\partial}{\partial \mu}(R^T K \dot{\mu}) \\ &= H_I^T R \frac{\partial}{\partial \mu}(R^T K \dot{\mu}) \end{aligned} \tag{20}$$

where $H_I \triangleq I_I \omega_I$ is the angular momentum vector expressed in the inertial reference frame. In the absence of external forces and moments, Lagrange's equations can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mu}}\right) - \left(\frac{\partial L}{\partial \mu}\right) = 0 \tag{21}$$

which after substitution of Eqs. (19) and (20) and the identity in Eq. (4) leads to

$$\dot{\boldsymbol{H}}_I^T \boldsymbol{K} = \boldsymbol{H}_I^T \boldsymbol{R}\left[\frac{\partial}{\partial \boldsymbol{\mu}}(\boldsymbol{R}^T \boldsymbol{K}\dot{\boldsymbol{\mu}}) - \boldsymbol{R}^T \dot{\boldsymbol{K}}\right] = 0 \tag{22}$$

Because the matrix $\boldsymbol{K}(\boldsymbol{\mu})$ is singular only for certain Euler angle coordinates, whereas the relation in Eq. (22) is true for all values of the Euler angles, we can conclude $\dot{\boldsymbol{H}}_I = 0$. This implies that the angular momentum of the rigid body is a conserved quantity.

Conclusion

The conservation of angular momentum of a free rigid body is a well-known fact. When the rigid body is described in terms of its Eulerian motion, however, this conservation law cannot be readily established. We present a new identity involving Euler angles and their rates. This identity provides a Noetherian perspective of free Eulerian motion and readily establishes the well-known fact that the angular momentum is conserved.

Acknowledgment

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Near-Optimal Low-Thrust Trajectories via Micro-Genetic Algorithms

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Introduction

GENETIC algorithms (GAs) are robust parameter optimization techniques based on the Darwinian concept of natural selection. Genetic algorithms have been successfully employed to determine both impulsive[1–3] and low-thrust orbit transfers.[4] The simplest form of a GA uses the three basic operators of reproduction, crossover, and mutation. Populations typically range from 30 to 200 individuals.[5] Here, the effectiveness of using micro-genetic algorithms (μGAs) to determine near-optimal low-thrust trajectories is investigated. Micro-GAs are GAs with populations typically fewer than 20 individuals. Schemes for using μGAs have been proposed[6] and have been shown to be more effective (i.e., have fewer function evaluations) than larger population GAs.[7,8]

The GA approach is extremely powerful at solving unconstrained optimization problems; however, the extension to constrained optimization remains a research issue. Two methods for handling constraints are studied. One technique enforces constraints through equality constraints appended to the objective function,[4] whereas the second approach constrains the problem via inequality constraints.[9]

Low-Thrust Problem Formulation

A control profile that minimizes propellant consumption while satisfying specified boundary conditions is to be determined. The planar dynamics in polar coordinates (r, θ) are given in Eqs. (1–4).[10] The radial velocity is denoted by u, tangential velocity v, gravitational constant of the attracting body μ, initial mass m_0, propellant consumption rate $\dot{m}$, and time t:

$$\dot{r} = u \tag{1}$$

$$\dot{\theta} = v/r \tag{2}$$

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin(\phi)}{m_0 - \dot{m}t} \tag{3}$$

$$\dot{v} = -\frac{uv}{r} + \frac{T\cos(\phi)}{m_0 - \dot{m}t} \tag{4}$$

The control profile is given by the thrust magnitude T and thrust direction ϕ. Because a GA is a parameter optimization technique, these continuous functions must be represented by a finite string of numbers. Therefore, the entire trajectory is broken into discrete segments. The GA then determines the thrust direction and magnitude at the beginning of each segment. The control is assumed constant over a segment, and the equations of motion are integrated. The total time required to complete the trajectory is fixed. In addition, the propulsion system contains a constant power source such as a nuclear electric engine and produces a constant thrust magnitude. The optimal thrust magnitude profile for these types of propulsion system typically consists of periods of burn arcs at maximum thrust and periods of coasting. Therefore, the thrust magnitude is modeled by a switching function modulating between on and off. The propellant consumption rate is supplied by Eq. (5), where the specific impulse of the propellant is denoted by I_{sp} and the Earth's gravity by g:

$$\dot{m} = T/gI_{sp} \tag{5}$$

GA Characteristics

Genetic algorithms were developed by John Holland and his students in the 1970s,[11] and a detailed description can be found in Ref. 5. For the GA used in this study, the parameters in the thrust profile (T and ϕ) are converted into a finite length string of binary characters. The engines are either on or off so that a single bit representation is sufficient for the thrust magnitude representation. The thrust direction resides between 0 and 360 deg. A 5-bit string is used, providing a resolution of 11.25 deg.

The simple GA used in this study is similar to that described in a previous low-thrust optimization study.[4] Tournament selection, single point crossover, and mutation were used. The μGA used tournament selection and uniform crossover, and these two operations continue until the population converges, typically every four to five generations. Convergence is considered achieved when fewer than 5% of the total number of bits of each individual are different from the best series of bits for that generation. Once this occurs, the best solution is copied over to the next generation, and the remaining population is again randomly recreated and the procedure repeats itself. Uniform crossover was chosen because it has been shown to be more robust than single point crossover for μGAs when treating problems with multiple local minimizing solutions.[8] No mutation is used in μGAs, because with the rapid convergence cycles mutations do not have time to evolve before a new random population is introduced.

Constraint Introduction

Genetic algorithms are well suited to solving unconstrained optimization problems of the form: Determine the vector $\boldsymbol{x}$ of dimension n to maximize the scalar function $f(\boldsymbol{x})$. However, when m equality constraints on $\boldsymbol{x}$ are introduced, a uniform procedure for posing the problem to GAs is not available. Here, two methods are investigated. In the following discussions, the m-dimensional vector of constraints is represented by Eq. (6):

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{0} \tag{6}$$

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